Topic: Product-to-sum identities

Question: Choose the correct expression.

Express $\cos(14\theta)\sin(-5\theta)$ in the form $a\left[\sin(b\theta)+\sin(c\theta)\right]$ where a and b are positive numbers and c is a negative number.

Answer choices:

A $(\cos 14\theta)(\sin -5\theta) = \frac{1}{2}\left[\sin(19\theta)+\sin(-5\theta)\right]$

B $(\cos 14\theta)(\sin -5\theta) = \frac{1}{2}\left[\sin(9\theta)+\sin(-19\theta)\right]$

C $(\cos 14\theta)(\sin -5\theta) = \frac{1}{2}\left[\sin(19\theta)+\sin(-14\theta)\right]$

D $(\cos 14\theta)(\sin -5\theta) = \frac{1}{2}\left[\sin(14\theta)+\sin(-9\theta)\right]$

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Solution: B

By one of the product-to-sum identities,

$$(\cos 14\theta)(\sin - 5\theta) = \frac{1}{2}\left[\sin(14\theta + (-5\theta)) - \sin(14\theta - (-5\theta))\right]$$

Simplifying:

$$(\cos 14\theta)(\sin - 5\theta) = \frac{1}{2}\left[\sin(14\theta - 5\theta) - \sin(14\theta + 5\theta)\right]$$

$$(\cos 14\theta)(\sin - 5\theta) = \frac{1}{2}\left[\sin(9\theta) - \sin(19\theta)\right]$$

By the odd identity for sine,

$$\sin(-19\theta) = -\sin 19\theta$$

Substituting $\sin(-19\theta)$ for $-\sin(19\theta)$ gives

$$(\cos 14\theta)(\sin - 5\theta) = \frac{1}{2}\left[\sin(9\theta) + \sin(-19\theta)\right]$$

This is in the prescribed form, with $a = 1/2$, $b = 9$, and $c = -19$.

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Topic: Product-to-sum identities

Question: Find the exact values.

Use product-to-sum identities to determine the values of $\sin(\pi/12)\cos(\pi/12)$ and $\sin(\pi/12)$.

Answer choices:

A $\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right)=\frac{1}{2}$ and $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{2-\sqrt{3}}}{2}$

B $\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right)=\frac{1}{4}$ and $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{2+\sqrt{3}}}{2}$

C $\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right)=\frac{1}{2}$ and $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{2+\sqrt{3}}}{2}$

D $\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right)=\frac{1}{4}$ and $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{2-\sqrt{3}}}{2}$

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Solution: D

By one of the product-to-sum identities,

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

Setting both θ and α to $\pi/12$, we have

$$\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right) = \frac{1}{2}\left[\sin\left(\frac{\pi}{12}+\frac{\pi}{12}\right)+\sin\left(\frac{\pi}{12}-\frac{\pi}{12}\right)\right]$$

Simplifying:

$$\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right) = \frac{1}{2}\left(\sin\frac{\pi}{6}+\sin 0\right)$$

We know that $\sin(\pi/6) = 1/2$ and $\sin 0 = 0$, so

$$\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right) = \frac{1}{2}\left(\frac{1}{2}+0\right) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

By another product-to-sum identity,

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

Setting both θ and α to $\pi/12$ gives

$$\left(\sin\frac{\pi}{12}\right)\left(\sin\frac{\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{12}-\frac{\pi}{12}\right)-\cos\left(\frac{\pi}{12}+\frac{\pi}{12}\right)\right]$$

Simplifying:

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$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(\cos 0 - \cos\frac{\pi}{6}\right)$$

We know that $\cos 0 = 1$ and $\cos(\pi/6) = \sqrt{3}/2$. Therefore,

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\left[\frac{2(1) - 1\left(\sqrt{3}\right)}{2}\right] = \frac{1}{2}\left(\frac{2-\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}$$

Since an angle of $\pi/12$ radians is in the first quadrant, we know that $\sin(\pi/12)$ is positive, so

$$\sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

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Topic: Product-to-sum identities

Question: Which of the following pairs is a solution of the equation?

$$\cos\theta\cos\alpha = -\frac{2+\sqrt{2}}{4}$$

Answer choices:

A $\left(\theta = \frac{17\pi}{8}, \alpha = \frac{25\pi}{8}\right)$

B $\left(\theta = \frac{11\pi}{8}, \alpha = \frac{7\pi}{8}\right)$

C $\left(\theta = \frac{5\pi}{8}, \alpha = \frac{7\pi}{8}\right)$

D $\left(\theta = \frac{9\pi}{8}, \alpha = \frac{11\pi}{8}\right)$

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Solution: A

By one of the product-to-sum identities,

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

Setting θ to $17\pi/8$ and α to $25\pi/8$, we have

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{17\pi}{8}+\frac{25\pi}{8}\right)+\cos\left(\frac{17\pi}{8}-\frac{25\pi}{8}\right)\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right) = \frac{1}{2}\left[\cos\frac{(17+25)\pi}{8}+\cos\frac{(17-25)\pi}{8}\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{42\pi}{8}\right)+\cos\left(\frac{-8\pi}{8}\right)\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{42\pi}{8}\right)+\cos(-\pi)\right]$$

Note that

$$\frac{42\pi}{8} = \frac{(40+2)\pi}{8} = 5\pi + \frac{2\pi}{8} = 5\pi + \frac{\pi}{4}$$

Therefore,

$$\cos\left(\frac{42\pi}{8}\right) = \cos\left(5\pi+\frac{\pi}{4}\right)$$

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Now an angle of 5π radians is on the negative x-axis, so an angle of $5\pi + (\pi/4)$ radians is halfway "between" the negative x-axis and the negative y-axis. Thus

$$\cos\left(5\pi + \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Also, we know that $\cos(-\pi) = -1$.

Substituting these results, we get

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = \frac{1}{2}\left[-\frac{\sqrt{2}}{2} + (-1)\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = \frac{1}{2}\left(-\frac{\sqrt{2}}{2} - \frac{2}{2}\right)$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = -\frac{1}{2}\left(\frac{2+\sqrt{2}}{2}\right)$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = -\frac{2+\sqrt{2}}{4}$$

To show that none of the other answer choices is correct, note that

$$-\frac{2+\sqrt{2}}{4} < 0$$

Therefore, it suffices to show that the sign of $\cos\theta\cos\alpha$ for each of the other three answer choices is positive.

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First, we'll look at answer choice B:

$$\left(\theta = \frac{11\pi}{8}, \alpha = \frac{7\pi}{8}\right)$$

Note that

$$\frac{11\pi}{8} = \frac{8\pi + 3\pi}{8} = \frac{8\pi}{8} + \frac{3\pi}{8} = \pi + \frac{3\pi}{8}$$

An angle of π radians is on the negative x-axis, and an angle of $3\pi/8$ radians is acute (it lies "between" 0 and $\pi/2$). Therefore, an angle of $11\pi/8$ radians is in the third quadrant, so its cosine is negative.

An angle of $7\pi/8$ radians is in the second quadrant (it lies "between" $\pi/2$ and π), so its cosine is also negative. Thus $\cos\theta\cos\alpha$ is positive.

Next, we'll consider answer choice C:

$$\left(\theta = \frac{5\pi}{8}, \alpha = \frac{7\pi}{8}\right)$$

An angle of $5\pi/8$ radians is in the second quadrant (it lies "between" $\pi/2$ and π), so its cosine is negative. We have already noted that an angle of $7\pi/8$ radians is in the second quadrant, hence that its cosine is negative as well. Therefore, $\cos\theta\cos\alpha$ is positive.

Finally, we'll look at answer choice D:

$$\left(\theta = \frac{9\pi}{8}, \alpha = \frac{11\pi}{8}\right)$$

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An angle of $9\pi/8$ radians lies "between" π and $11\pi/8$. Now we have already found that an angle of $11\pi/8$ radians is in the third quadrant. Thus $9\pi/8$ is also in the third quadrant. From this it follows that the cosines of both $9\pi/8$ and $11\pi/8$ are negative, hence that $\cos\theta\cos\alpha$ is positive.

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